

Pseudoinstantons in metric-affine field theory

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Abstract

In abstract Yang–Mills theory the standard instanton construction relies on the Hodge star having real eigenvalues which makes it inapplicable in the Lorentzian case. We show that for the affine connection an instanton-type construction can be carried out in the Lorentzian setting. The Lorentzian analogue of an instanton is a spacetime whose connection is metric compatible and Riemann curvature irreducible (“pseudoinstanton”). We suggest a metric-affine action which is a natural generalization of the Yang–Mills action and for which pseudoinstantons are stationary points. We show that a spacetime with a Ricci flat Levi-Civita connection is a pseudoinstanton, so the vacuum Einstein equation is a special case of our theory. We also find another pseudoinstanton which is a wave of torsion in Minkowski space. Analysis of the latter solution indicates the possibility of using it as a model for the neutrino.

KEY WORDS: Yang–Mills equation; instanton; gravity; torsion; neutrino

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1 Statement of the problem

We consider spacetime to be a connected real oriented 4-manifold M equipped with a Lorentzian metric g and an affine connection Γ . The 10 independent components of the metric tensor $g_{\mu\nu}$ and the 64 connection coefficients $\Gamma^\lambda_{\mu\nu}$ are the unknowns of our theory, as is the manifold M itself.

It is known (see Appendix B.4 in [1] as well as Appendix A in our paper) that at each point $x \in M$ the vector space of (real) Riemann curvatures decomposes under the Lorentz group into a direct sum of eleven invariant subspaces which are irreducible and mutually orthogonal. Given a Riemann curvature R we will denote by $R^{(j)}$, $j = 1, \dots, 11$, its irreducible pieces.

The natural inner product on Riemann curvatures is

$$(R, Q) := \int R^\kappa_{\lambda\mu\nu} Q^\lambda{}_\kappa{}^{\mu\nu}.$$

We denote $\|R\|^2 := (R, R)$. Of course, our inner product is indefinite, so $\|R\|^2$ does not have a particular sign and we cannot attribute a meaning to $\|R\|$ itself.

We define our action as

$$S := \sum_{j=1}^{11} c_j \|R^{(j)}\|^2, \quad (1)$$

where the c_j 's are real constants. Note the analogy between formula (1) and the potential energy of an isotropic elastic body, see formulae (4.2), (4.3) in [2]. The only difference is that in the theory of elasticity the field strength is the deformation tensor (rank 2 symmetric tensor) rather than Riemann curvature, and it has two irreducible pieces (shear and hydrostatic compression) rather than eleven. Note also that the idea of using an action of the type (1) goes back to Weyl who argued at the end of his 1919 paper [3] that the most natural gravitational action should be quadratic in curvature and involve its irreducible pieces as separate terms. Weyl wrote: "I intend to pursue the consequences of this action principle in a continuation of this paper". (Translation by G. Friesecke.) It is regrettable that Weyl never carried out this analysis.

Variation of the action (1) with respect to the metric g and the connection Γ produces Euler–Lagrange equations which we will write symbolically as

$$\partial S / \partial g = 0, \quad (2)$$

$$\partial S / \partial \Gamma = 0. \quad (3)$$

Our objective is the study of the combined system (2), (3). This is a system of $10 + 64$ real nonlinear partial differential equations with $10 + 64$ real unknowns.

Remark 1.1 *It is easy to see that the action (1) is conformally invariant, i.e., it does not change if we perform a Weyl rescaling of the metric $g \rightarrow e^{2f}g$, $f : M \rightarrow \mathbb{R}$, without changing the connection Γ (here it is important that in the metric-affine setting the metric and the connection lead a separate existence). Therefore, the number of independent equations in (2) is not 10 but 9.*

Following Eisenhart [4] we call a spacetime *Riemannian* if its connection is Levi-Civita (i.e., $\Gamma^\lambda_{\mu\nu} = \{\lambda_{\mu\nu}\}$) and *non-Riemannian* otherwise. Here “Riemannian” does not imply the positivity of the metric, the latter being assumed to be Lorentzian throughout the paper.

In the special case

$$c_1 = \dots = c_{11} = 1 \quad (4)$$

the functional (1) becomes $\|R\|^2$. This is the Yang–Mills action for the affine connection, and equation (3) is the corresponding Yang–Mills equation. The latter was analyzed by Yang [5]. Yang was looking for Riemannian solutions, so he specialized equation (3) to the Levi-Civita connection and arrived at the equation

$$\nabla_\lambda Ric_{\kappa\mu} - \nabla_\kappa Ric_{\lambda\mu} = 0. \quad (5)$$

Here “specialization” means that one sets $\Gamma^\lambda_{\mu\nu} = \{\lambda_{\mu\nu}\}$ *after* the variation in Γ is carried out. An immediate consequence of equation (5) is the fact that Einstein spaces satisfy the Yang–Mills equation (3).

A number of other authors observed, still under the assumption (4), that a much stronger result is true: Einstein spaces satisfy both equations (2) and (3). An elegant explanation of this fact in terms of double duality was given by Mielke [6]. Mielke’s paper was written for the case of a positive metric but the result remains true for the Lorentzian case, the only difference being that one has to change signs in double duality formulae. We shall therefore refer to the special case (4) of the model (2), (3) as the *Yang–Mielke theory of gravity*.

Apart from [5, 6] there have been numerous other publications on the subject, with many authors independently rediscovering known results. One can get an idea of the historical development of the Yang–Mielke theory of gravity from [7, 8, 9, 10, 11, 12, 13, 14, 15]. Of these publications the most remarkable is the Mathematical Review [8]: the author of the review noticed a fact missed in the paper under review [7], namely, that Einstein spaces are stationary points of the Yang–Mills action with respect to the variation of both the metric and the connection, a fact repeatedly rediscovered in later years.

Our aim is to develop the Yang–Mielke theory of gravity by

- dropping the requirement (4),
- looking for Riemannian solutions other than Einstein spaces, and
- looking for non-Riemannian solutions.

2 Main result

The following definition is crucial in our construction.

Definition 2.1 *We call a spacetime a pseudoinstanton if its connection is metric compatible and Riemann curvature irreducible.*

Here irreducibility of Riemann curvature means that of the eleven $R^{(j)}$'s all except one are identically zero. In fact, metric compatibility cuts the number of possible irreducible pieces to six. Explicit formulae for the latter are given at the end of Appendix A.

Definition 2.1 is motivated by the analogy with abstract Yang–Mills theory in Euclidean space, see Sections 3 and 4 of Chapter 1 in [16]. Indeed, the notion of an instanton is based on the decomposition of the vector space of curvatures into two subspaces which are invariant under the action of the orthogonal group on the external indices. (We call the Lie algebra indices of curvature internal, and the remaining ones external.) The case of the affine connection is special in that the internal and external indices have the same nature, so it is logical to apply (pseudo)orthogonal transformations to the whole rank 4 tensor. This leads to a richer algebraic structure.

Our main result is

Theorem 2.1 *A pseudoinstanton is a solution of the problem (2), (3).*

In Section 4 we prove Theorem 2.1, and in the remainder of the paper we use this theorem for constructing families of solutions of the system (2), (3).

3 Notation

Our notation follows [17]. In particular, we denote local coordinates by x^μ , $\mu = 0, 1, 2, 3$, and write $\partial_\mu := \partial/\partial x^\mu$. We define the covariant derivative of a vector function as $\nabla_\mu v^\lambda := \partial_\mu v^\lambda + \Gamma^\lambda_{\mu\nu} v^\nu$, torsion as $T^\lambda_{\mu\nu} := \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}$, contortion as

$$K^\lambda_{\mu\nu} := \frac{1}{2}(T^\lambda_{\mu\nu} + T^\lambda_{\nu\mu} + T^\lambda_{\mu\lambda}) \quad (6)$$

(see formula (7.35) in [18]), Riemann curvature as

$$R^\kappa_{\lambda\mu\nu} := \partial_\mu \Gamma^\kappa_{\nu\lambda} - \partial_\nu \Gamma^\kappa_{\mu\lambda} + \Gamma^\kappa_{\mu\eta} \Gamma^\eta_{\nu\lambda} - \Gamma^\kappa_{\nu\eta} \Gamma^\eta_{\mu\lambda}, \quad (7)$$

Ricci curvature as $Ric_{\lambda\nu} := R^\kappa_{\lambda\kappa\nu}$, scalar curvature as $\mathcal{R} := Ric^\lambda_\lambda$, and trace free Ricci curvature as $\mathcal{Ric}_{\lambda\nu} := Ric_{\lambda\nu} - \frac{1}{4}g_{\lambda\nu}\mathcal{R}$. We denote Weyl curvature by $\mathcal{W} = R^{(3)}$.

It is easy to see that contortion has the antisymmetry property $K_{\lambda\mu\nu} = -K_{\nu\mu\lambda}$ and that

$$T^\lambda_{\mu\nu} = K^\lambda_{\mu\nu} - K^\lambda_{\nu\mu}. \quad (8)$$

Formulae (6), (8) allow us to express torsion and contortion via one another.

A connection is said to be metric compatible if $\nabla_\lambda g_{\mu\nu} \equiv 0$. A metric compatible connection is uniquely determined by metric and torsion or metric and contortion, see Section 7.2.6 in [18] for details. In the metric compatible case contortion can be written as

$$K^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\}, \quad (9)$$

where

$$\left\{ \begin{smallmatrix} \lambda \\ \mu\nu \end{smallmatrix} \right\} := \frac{1}{2} g^{\lambda\kappa} (\partial_\mu g_{\nu\kappa} + \partial_\nu g_{\mu\kappa} - \partial_\kappa g_{\mu\nu}) \quad (10)$$

is the Christoffel symbol.

The choice between using torsion and using contortion is a matter of taste. When working with metric compatible connections using contortion is somewhat more convenient because formula (9) is so simple and natural.

Given a scalar function $f : M \rightarrow \mathbb{R}$ we write for brevity

$$\int f := \int_M f \sqrt{|\det g|} \, dx^0 dx^1 dx^2 dx^3, \quad \det g := \det(g_{\mu\nu}).$$

Throughout the paper we work only in coordinate systems with positive orientation. Moreover, when we restrict our consideration to Minkowski space we assume that our coordinate frame is obtained from a given reference frame by a proper Lorentz transformation.

We define the action of the Hodge star on a rank q antisymmetric tensor as

$$(*Q)_{\mu_{q+1}\dots\mu_4} := (q!)^{-1} \sqrt{|\det g|} \, Q^{\mu_1\dots\mu_q} \varepsilon_{\mu_1\dots\mu_4}, \quad (11)$$

where ε is the totally antisymmetric quantity, $\varepsilon_{0123} := +1$.

4 Proof of the main theorem

This section is devoted to the proof of Theorem 2.1.

Let us first examine what happens when we fix the metric and vary the connection. The explicit formula for the variation of the action δS resulting from the variation of the connection $\delta\Gamma$ is

$$\begin{aligned} \delta S &= 2 \sum_{j=1}^{11} c_j \int \text{tr}((R^{(j)})^{\mu\nu} (\delta R^{(j)})_{\mu\nu}) = 2 \sum_{j=1}^{11} c_j \int \text{tr}((R^{(j)})^{\mu\nu} (\delta R)_{\mu\nu}) \\ &= 4 \sum_{j=1}^{11} c_j \int \text{tr}((\delta_{\text{YM}} R^{(j)})^\mu (\delta\Gamma)_\mu) \end{aligned}$$

where δ_{YM} is the Yang–Mills divergence,

$$(\delta_{\text{YM}} R)^\mu := \frac{1}{\sqrt{|\det g|}} (\partial_\nu + [\Gamma_\nu, \cdot]) (\sqrt{|\det g|} R^{\mu\nu}).$$

Here, as in [17, 19], we use matrix notation to hide the two internal indices. We start our variation from a spacetime with a metric compatible connection (see Definition 2.1) and this fact has important consequences. We have $R^{(j)} \equiv 0$ for $j = 7, \dots, 11$ (see Appendix A for details). The remaining curvatures $R^{(j)}$, $j = 1, \dots, 6$, are antisymmetric in the internal indices and, moreover,

the action of the Yang–Mills divergence preserves this property. (This is, of course, a consequence of the fact that antisymmetric rank 2 tensors form a subalgebra within the general Lie algebra of rank 2 tensors.) Therefore, in order to prove that we have a stationary point with respect to arbitrary variations of the connection it is necessary and sufficient to prove that we have a stationary point with respect to variations of the connection which are antisymmetric in the internal indices, i.e., variations satisfying $g_{\kappa\lambda}(\delta\Gamma)^\lambda_{\mu\nu} + g_{\nu\lambda}(\delta\Gamma)^\lambda_{\mu\kappa} = 0$. But this means that it is necessary and sufficient to prove that we have a stationary point with respect to variations of the connection which preserve metric compatibility. So further on in this section we work with metric compatible connections only.

We start our variation from a spacetime which is a pseudoinstanton, therefore for some $l \in \{1, \dots, 6\}$ we have

$$R^{(j)} \equiv 0, \quad \forall j \neq l. \quad (12)$$

Let us rewrite formula (1) as

$$S = c_l \|R\|^2 + \sum_{j=1}^{11} (c_j - c_l) \|R^{(j)}\|^2 \quad (13)$$

and vary the metric and the connection. Formulae (12), (13) imply that $\delta S = c_l \delta(\|R\|^2)$. So in order to prove Theorem 2.1 it is sufficient to show that our pseudoinstanton is a stationary point of the Yang–Mills action $\|R\|^2$.

The remainder of the proof is an adaptation of Mielke’s argument [6].

Let us first assume for simplicity that our manifold is compact. For compact manifolds with metric compatible connections we have the identity

$$\|R\|^2 = \frac{1}{2} \|R \mp {}^*R^*\|^2 \pm (R, {}^*R^*), \quad (14)$$

where ${}^*R^*$ is defined in accordance with formulae (49), (50), (55). It is known that $(R, {}^*R^*)$ is a topological invariant: it is, up to a normalizing factor, the Euler number of the manifold, see Section 5 of Chapter XII and Note 20 in [20]. (Actually, the Euler number of a compact Lorentzian manifold can only be zero, see [21], p. 207.) So it remains to show that a pseudoinstanton is a stationary point of the functional $\|R \mp {}^*R^*\|^2$. But this is a consequence of the fact that irreducibility of Riemann curvature implies ${}^*R^* = \pm R$, see Appendix A for details.

In the case of a noncompact manifold one should understand the identity (14) in the Euler–Lagrange sense. The statement that $(R, {}^*R^*)$ is a topological invariant means now that this functional generates zero Euler–Lagrange terms. Euler–Lagrange arguments are purely local and the fact that $(R, {}^*R^*)$ does not contribute to the Euler–Lagrange equations is unrelated to the compactness or noncompactness of the manifold.

The proof of Theorem 2.1 is complete.

5 The Weyl pseudoinstanton

In order to start constructing pseudoinstantons we need to choose the irreducible subspace $\mathbf{R}^{(l)}$ into which we will attempt to fit our Riemann curvature. We choose to look first for pseudoinstantons in the subspace of Weyl curvatures $\mathbf{R}^{(3)}$. This choice is motivated by the observation that of the six possible subspaces $\mathbf{R}^{(l)}$, $l = 1, \dots, 6$, generated by a metric compatible connection the subspace $\mathbf{R}^{(3)}$ has the highest dimension, so it will be easier to fit our curvature into this subspace.

Let R be the Riemann curvature generated by a metric compatible connection. Then $R \in \mathbf{R}^{(3)}$ if and only if

$$R^T = R, \quad (15)$$

$$Ric = 0, \quad (16)$$

$$\varepsilon^{\kappa\lambda\mu\nu} R_{\kappa\lambda\mu\nu} = 0, \quad (17)$$

where transposition is defined in accordance with formula (48). Equation (16) in the above system can, of course, be replaced by the pair of equations $*R^* = -R$, $\mathcal{R} = 0$.

Definition 5.1 *We call a metric compatible solution of the system (15)–(17) a Weyl pseudoinstanton.*

This terminology is motivated by the fact that such Riemann curvatures are purely Weyl.

In the next two sections we construct explicitly two families of Weyl pseudoinstantons.

6 The vacuum Einstein equation

In this section we look for Riemannian Weyl pseudoinstantons, that is, for Weyl pseudoinstantons with zero torsion. In this case the connection is Levi-Civita and equations (15), (17) are automatically satisfied. This leaves us with equation (16) which is the vacuum Einstein equation.

Thus, the vacuum Einstein equation is simply the explicit description of a Riemannian Weyl pseudoinstanton.

7 Torsion waves

In this section we look for non-Riemannian Weyl pseudoinstantons, that is, for Weyl pseudoinstantons with non-zero torsion. Throughout the section we work in Minkowski space which we define as a real 4-manifold with global coordinate system (x^0, x^1, x^2, x^3) and metric

$$g_{\mu\nu} = \text{diag}(+1, -1, -1, -1). \quad (18)$$

Note that our definition of Minkowski space specifies the manifold M and the metric g , but does not specify the connection Γ .

The construction we are about to carry out is, in a sense, the opposite of what we did in the previous section: in Section 6 we looked for Weyl pseudoinstantons with non-trivial metric and zero torsion, whereas now we will be looking for Weyl pseudoinstantons with trivial metric (18) and non-zero torsion. It is important to emphasize that the fact that the metric is constant does not imply that curvature is zero because the connection coefficients appearing in (7) are not necessarily Christoffel symbols.

Formulae (9), (10), (18) imply $K^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu}$, so in the Minkowski setting the system (15)–(17) is a system of first order partial differential equations for the unknown contortion. There are two difficulties associated with this system. Firstly, it is overdetermined: the number of independent equations is $15 + 10 + 1$ whereas the number of unknowns (independent components of the contortion tensor) is only 24. Secondly, it has a quadratic nonlinearity resulting from the commutator in the formula for Riemann curvature (last two terms in the RHS of formula (7)).

The second difficulty is fundamental, however it can be overcome by means of the *linearization ansatz* suggested in [17, 19]. Namely, one seeks the unknown contortion in the form $K^\lambda_{\mu\nu} = \text{Re}(v_\mu L^\lambda_\nu)$ where v is a complex-valued vector function and $L \neq 0$ is a constant complex antisymmetric tensor satisfying $*L = \pm iL$. Then the nonlinear system (15)–(17) turns into a *linear* system for the vector function v . The coefficients of this linear system of partial differential equations depend on the tensor L as a parameter, and this parameter dependence is also linear.

It is interesting that the idea of seeking the unknown rank 3 tensor in the form of a product of a vector and a rank 2 tensor (“separation of indices”) goes back to Lanczos, see formula (XI.1) in his paper [22]. Unfortunately, Lanczos did not develop this idea. He restricted his analysis to the following observation: “Such solutions cannot be studied on the basis of purely linear operators . . . Hence they are outside the limits of the present investigation.”

Explicit calculations [17, 19] produce a Weyl pseudoinstanton which can be written down in the following compact form. This metric compatible spacetime is characterized by metric (18) and torsion

$$T^\lambda_{\mu\nu} = \frac{1}{2} \text{Re}(u^\lambda (du)_{\mu\nu}) \quad (19)$$

where u is a non-trivial plane wave solution of the polarized Maxwell equation

$$*du = \pm i du. \quad (20)$$

Here d is the operator of exterior differentiation, u is a complex-valued vector function, “plane wave” means that $u(x) = w e^{-ik \cdot x}$ where w is a constant complex vector and k is a constant real vector, and “non-trivial” means that $du \neq 0$.

Let us stress that the spacetime (18)–(20) is a solution of the full nonlinear system (15)–(17), and, consequently, a solution of the full nonlinear system (2), (3).

A detailed analysis of the solution (18)–(20) carried out in [17, 19] shows that it may be interpreted as the neutrino. This interpretation is based on the examination of the corresponding Riemann curvature

$$R_{\kappa\lambda\mu\nu} = \text{Re}((du)_{\kappa\lambda}(du)_{\mu\nu}). \quad (21)$$

Clearly, $(du)_{\kappa\lambda}(du)_{\mu\nu}$ is a factorized Weyl curvature, which according to Lemma C.1 makes it equivalent to a spinor $\xi = \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix}$ or $\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$. It turns out that this spinor function satisfies the appropriate half of Weyl's equation $\gamma^\mu \partial_\mu \psi = 0$, that is,

$$\partial_0 \xi + (\sigma^1 \partial_1 + \sigma^2 \partial_2 + \sigma^3 \partial_3) \xi = 0$$

or

$$\partial_0 \eta - (\sigma^1 \partial_1 + \sigma^2 \partial_2 + \sigma^3 \partial_3) \eta = 0.$$

Remark 7.1 *It is known (see, for example, [23]) that in a non-Riemannian spacetime with metric compatible connection Weyl's equation has an additional term with torsion. However, this additional term involves only the axial component of torsion which in our case is zero. See Appendix B.2 in [1] or Appendix B in our paper for irreducible decomposition of torsion.*

In interpreting the solution (18)–(20) as the neutrino we chose to deal with curvature rather than torsion because curvature is an accepted physical observable. If we also accept torsion as a physical observable then the situation changes. Given a plane wave solution u of the polarized Maxwell equation (20) we can always add to it the gradient of a scalar plane wave, which changes torsion (19) but does not change curvature (21). Thus, different torsions can generate the same curvature, and having accepted torsion as a physical observable we have to treat these solutions as different particles. This might explain the subtle difference between the electron, muon and tau neutrinos.

Recently there has been a series of publications [24, 25, 26, 27, 28] in which the authors constructed other types of torsion waves. Our torsion waves (18)–(20) are fundamentally different from those in [24, 25, 26, 27, 28]. The differences are as follows.

- The action in [24, 25, 26, 27, 28] is more general in that it contains terms with torsion and nonmetricity. However, the solutions found in these publications become Riemannian for our action (1).
- The metric in [24, 25, 26, 27, 28] is non-constant and the connection is not metric compatible, whereas our metric is constant and connection is metric compatible.

- The torsion in [24, 25, 26, 27, 28] is purely trace, whereas ours is purely tensor.
- Our torsion and Riemann curvature are monochromatic plane waves, i.e.,

$$\begin{aligned} T(x) &= T' \cos(k \cdot x) + T'' \sin(k \cdot x), \\ R(x) &= R' \cos(k \cdot x) + R'' \sin(k \cdot x), \end{aligned}$$

where T' , T'' , R' , and R'' are constant real tensors. In [24, 25, 26, 27, 28] torsion and curvature are more complex.

The most important feature of our torsion wave (18)–(20) is the fact that

$$R \in \mathbf{R}^{(1)} \oplus \mathbf{R}^{(2)} \oplus \mathbf{R}^{(3)}. \quad (22)$$

The RHS of formula (22) is the space of Riemann curvatures generated by Levi-Civita connections. Formula (22) means that the curvature generated by our torsion wave has all the symmetry properties of the usual curvature from Riemannian geometry. Therefore, in observing such a torsion wave we might not interpret it as torsion at all and believe that we live in a Riemannian universe.

The torsion waves in [24, 25, 26, 27, 28] do not possess the property (22).

8 The Ricci pseudoinstanton

Let R be the Riemann curvature generated by a metric compatible connection. Then $R \in \mathbf{R}^{(1)}$ if and only if equations (15), (17), and

$$\mathcal{R} = 0, \quad (23)$$

$$\mathcal{W} = 0 \quad (24)$$

are satisfied. The last three equations in the system (15), (17), (23), (24) can, of course, be replaced by the equation $*R* = R$.

Definition 8.1 *We call a metric compatible solution of the system (15), (17), (23), (24) a Ricci pseudoinstanton.*

This terminology is motivated by the fact that such Riemann curvatures are completely determined by the trace free Ricci tensor.

We cannot produce torsion wave solutions of the system (15), (17), (23), (24). More precisely, our linearization ansatz [17, 19] when applied to this system does not produce non-trivial ($R \neq 0$) solutions. There are, however, Riemannian solutions.

Definition 8.2 *We call a Riemannian spacetime a Thompson space if its scalar and Weyl curvatures are zero.*

Thompson noticed [9] that such spaces satisfy equation (3). Later Fairchild addressed the question whether Thompson spaces satisfy equation (2). He first thought [12] that they do not, but in a subsequent erratum [13] concluded that Thompson spaces do indeed satisfy equation (2). Thompson and Fairchild carried out their analysis for the Yang–Mills case (4) but the result remains true for arbitrary weights c_j because Thompson spaces are Ricci pseudoinstantons.

The physical meaning of Thompson spaces is unclear. It has been suggested by Thompson [9, 11], Pavelle [10], and Fairchild [12, 13] that these are nonphysical solutions.

It is worth noting that in appropriate local coordinates the metric of a Thompson space can be written as $g_{\mu\nu} = e^{2f} \text{diag}(+1, -1, -1, -1)$ where f is a real scalar function satisfying $\square f + \|\text{grad} f\|^2 = 0$. Here $\square := \partial_\mu \partial^\mu$, $(\text{grad} f)_\mu := \partial_\mu f$, $\|v\|^2 := v_\mu v^\mu$, and the raising of indices is performed with respect to the Minkowski metric (18). Our problem (2), (3) is conformally invariant (see Remark 1.1), therefore the natural thing to do is to rescale the metric and view such a solution as a scalar field on a manifold with Minkowski metric.

9 Einstein spaces

In this section we look for solutions of the system (2), (3) which are not pseudoinstantons. As this is an exceptionally difficult mathematical problem we restrict our search to Riemannian spacetimes.

Lengthy but straightforward calculations give the following explicit representation for equations (2), (3):

$$(c_1 + c_3) \mathcal{W}^{\kappa\lambda\mu\nu} \mathcal{R}ic_{\kappa\mu} + \frac{c_1 + c_2}{6} \mathcal{R} \mathcal{R}ic^{\lambda\nu} = 0, \quad (25)$$

$$(c_1 + c_3)(\nabla_\lambda \mathcal{R}ic_{\kappa\mu} - \nabla_\kappa \mathcal{R}ic_{\lambda\mu}) + \left(\frac{c_1}{4} + \frac{c_2}{6} + \frac{c_3}{12}\right)(g_{\kappa\mu} \partial_\lambda \mathcal{R} - g_{\lambda\mu} \partial_\kappa \mathcal{R}) = 0. \quad (26)$$

Weyl curvature has been excluded from equation (26) by means of the Bianchi identity. Note that this “trick” does not work for a general affine connection, nor does it work for a general metric compatible connection.

Definition 9.1 *We call a Riemannian spacetime an Einstein space if its Ricci curvature and metric are related as*

$$\mathcal{R}ic = \Lambda g \quad (27)$$

where Λ is some real “cosmological” constant.

Alternatively, an Einstein space can be defined as a Riemannian spacetime with

$$\mathcal{R}ic = 0. \quad (28)$$

Formula (28) and the contracted Bianchi identity imply

$$\mathcal{R} = 4\Lambda \quad (29)$$

with some constant Λ . The pair of conditions (28), (29) is, of course, equivalent to condition (27).

Clearly, equations (28), (29) imply equations (25), (26), so Einstein spaces are solutions of our problem (2), (3). Thus, our model with arbitrary weights c_j inherits the main feature of the Yang–Mielke theory of gravity.

Remark 9.1 *The fact that Einstein spaces are solutions of the system (2), (3) in the case of arbitrary weights c_j was already known to Buchdahl [8]. Buchdahl's review appears to have escaped the attention of subsequent researchers in the subject area.*

10 Discussion of the Riemannian case

We have found two families of Riemannian solutions to our problem (2), (3), namely, Thompson and Einstein spaces. Thompson spaces fit into our pseudo-instanton scheme whereas Einstein spaces do not (their Riemann curvature has, in general, two non-trivial irreducible pieces). It is natural to attempt to explain why Einstein spaces are solutions without having to write down explicitly the Euler–Lagrange equations. The explanation is as follows. In order to adapt the arguments from Section 4 to the case of an Einstein space it is necessary and sufficient to show that

$$(c_2 - c_3) \partial(\|R^{(2)}\|^2)/\partial g = 0, \quad (30)$$

$$(c_2 - c_3) \partial(\|R^{(2)}\|^2)/\partial \Gamma = 0. \quad (31)$$

The case $c_2 = c_3$ is trivial, so further on in this paragraph we assume $c_2 \neq c_3$. The reason why equations (30), (31) are satisfied for an Einstein space is that the irreducible piece $R^{(2)}$ has a very simple structure. We have $\|R^{(2)}\|^2 = -\frac{1}{6} \int \mathcal{R}^2$, and elementary calculations show that the system (30), (31) is equivalent to

$$\mathcal{R} \mathcal{R}ic = 0, \quad (32)$$

$$\partial \mathcal{R} = 0. \quad (33)$$

These equations are clearly satisfied under the conditions (28), (29).

In establishing that Thompson spaces are solutions of the system (2), (3) we relied on our general pseudoinstanton construction, without analyzing the actual Euler–Lagrange equations which in the Riemannian case have the explicit representation (25), (26). It may be worrying that the inspection of equation (26) does not immediately confirm that for a Thompson space this equation is satisfied. These fears are laid to rest if one rewrites equation (26) in equivalent form excluding the trace free Ricci curvature by means of the identity

$$\nabla_\lambda \mathcal{R}ic_{\kappa\mu} - \nabla_\kappa \mathcal{R}ic_{\lambda\mu} = -\frac{1}{12}(g_{\kappa\mu} \partial_\lambda \mathcal{R} - g_{\lambda\mu} \partial_\kappa \mathcal{R}) + 2\nabla_\nu \mathcal{W}_{\kappa\lambda\mu}{}^\nu \quad (34)$$

(consequence of the Bianchi identity). This turns equation (26) into

$$(c_1 + c_3)\nabla_\nu \mathcal{W}_{\kappa\lambda\mu}{}^\nu + \frac{c_1 + c_2}{12}(g_{\kappa\mu}\partial_\lambda \mathcal{R} - g_{\lambda\mu}\partial_\kappa \mathcal{R}) = 0$$

which is clearly satisfied under the conditions (23), (24).

An interesting feature of the system (25), (26) is that it does not contain the parameters c_1 , c_2 , c_3 separately, only their combinations $c_1 + c_2$ and $c_1 + c_3$. This warrants an explanation which goes as follows. We know (see Section 4) that for spacetimes with metric compatible connections the expression

$$\|R + {}^*R^*\|^2 - \|R - {}^*R^*\|^2$$

is a topological invariant. Therefore,

$$\delta(\|R + {}^*R^*\|^2) = \delta(\|R - {}^*R^*\|^2).$$

If we start our variation from a Riemannian spacetime then the latter formula becomes

$$\delta(\|R^{(1)}\|^2) = \delta(\|R^{(2)}\|^2) + \delta(\|R^{(3)}\|^2). \quad (35)$$

Formula (35) was written under the assumption that variation preserves metric compatibility, however arguments presented in the first paragraph of Section 4 show that it remains true for arbitrary variations. Formulae (1), (35) imply

$$\delta S = (c_1 + c_2)\delta(\|R^{(2)}\|^2) + (c_1 + c_3)\delta(\|R^{(3)}\|^2)$$

which explains why the resulting Euler-Lagrange equations contain only the combinations of weights $c_1 + c_2$ and $c_1 + c_3$.

Finally, let us give a simple characterization of Thompson and Einstein spaces. It is easy to see that these spaces are of opposite double duality: we have

$${}^*R^* = +R, \quad (36)$$

$${}^*R^* = -R \quad (37)$$

for Thompson and Einstein spaces respectively. Moreover, a Riemannian spacetime is a Thompson space if and only if it satisfies condition (36), and an Einstein space if and only if it satisfies condition (37).

11 Uniqueness

We have constructed in total three families of solutions to our problem (2), (3): torsion waves in Minkowski space (Section 7), Thompson spaces (Section 8), and Einstein spaces (Section 9). The question we are about to address is whether these three families are *all* the solutions of the problem (2), (3) within suitable classes of solutions. As questions of uniqueness in metric-affine field theory are notoriously difficult we will be forced to argue mostly at the level of conjectures.

Conjecture 11.1 *For generic weights c_j torsion waves constructed in Section 7 are the only solutions of our problem (2), (3) among connections of the type*

$$\Gamma(x) = \Gamma' \cos(k \cdot x) + \Gamma'' \sin(k \cdot x), \quad k \neq 0, \quad R(x) \not\equiv 0$$

in Minkowski space.

Conjecture 11.1 is motivated by the fact that in the Minkowski setting (18) the system (2), (3) is heavily overdetermined: it is a system of $9 + 64$ equations with only 64 unknowns.

The construction carried out in Section 7 is effectively based on the use of hidden symmetries of our problem, and there is no obvious way of generalizing it unless there are some additional symmetries due to a special choice of weights c_j . An example of such a special choice is the Yang–Mills case (4). Theorem 1 from [17] establishes that in this case the problem (2), (3) has a wider family of torsion wave solutions than those described in Section 7. Thus, the Yang–Mills case (4) is not generic in the sense of Conjecture 11.1.

The fact that in the Yang–Mills case (4) the problem (2), (3) has too many torsion wave solutions leads to serious difficulties. In [17] we were unable to attribute a physical interpretation to all these solutions, and, in order to reduce the number of solutions, were forced to introduce the vacuum Einstein equation (16) as an additional equation in our model. In the current paper this difficulty has been overcome by switching from the Yang–Mills action $\|R\|^2$ to the action (1) with arbitrary weights c_j . In this case the only general tool at our disposal is the pseudoinstanton construction which naturally leads to the vacuum Einstein equation (16).

Conjecture 11.2 *For generic weights c_j Thompson and Einstein spaces are the only Riemannian solutions of our problem (2), (3).*

Conjecture 11.2 is motivated by the following arguments. Suppose

$$c_1 + c_2 \neq 0, \quad c_1 + c_3 \neq 0. \quad (38)$$

Then the system (25), (26) is equivalent to the following system: equation (29) with some constant Λ and equations

$$\mathcal{W}^{\kappa\lambda\mu\nu} \mathcal{R}ic_{\kappa\mu} + c \mathcal{R} \mathcal{R}ic^{\lambda\nu} = 0, \quad (39)$$

$$\nabla_\lambda \mathcal{R}ic_{\kappa\mu} - \nabla_\kappa \mathcal{R}ic_{\lambda\mu} = 0, \quad (40)$$

where

$$c := \frac{c_1 + c_2}{6(c_1 + c_3)} \neq 0 \quad (41)$$

is a dimensionless parameter. (Of course, by virtue of the identity (34) the pair of equations (29) and (40) is equivalent to the pair of equations (29) and

$\nabla_\nu \mathcal{W}_{\kappa\lambda\mu}{}^\nu = 0$.) The constant Λ in (29) is either 0 or it scales to ± 1 , so the system (29), (39), (40) effectively contains only one free parameter, c . The number of independent equations in (29), (39), (40) is $1 + 9 + 16$ whereas the number of unknowns (independent components of the metric tensor) is only 10. It is hard to imagine how this overdetermined system can have solutions without the symmetry (36) or (37), except for some special values of the parameter c .

The search for special values of c for which our problem has more Riemannian solutions than expected is similar to the Cosserat problem in the theory of elasticity, which is the study of the elasticity operator with Poisson's ratio treated as the spectral parameter; see [29] and the extensive bibliographic list therein for details. In our case the parameter (41) plays the role of Poisson's ratio.

As an illustration let us consider the $2 + 2$ decomposable case when our 4-manifold is the product of two Riemannian 2-manifolds, see [11] or [30] for details. Straightforward calculations establish the following result within this class of solutions:

- if $c \neq -\frac{1}{3}$ then Thompson and Einstein spaces are the only solutions of the problem (2), (3), whereas
- if $c = -\frac{1}{3}$ then Thompson and Einstein spaces are not the only solutions of the problem (2), (3).

We see that the case $c = -\frac{1}{3}$ is not generic in the sense of Conjecture 11.2. It is interesting that condition (4) implies condition (38) as well as $c \neq -\frac{1}{3}$, so it may be that the Yang–Mills case is generic in the sense of Conjecture 11.2.

One should have in mind that the problem of uniqueness is very delicate even in the Yang–Mills case (4) and even within the class of Riemannian solutions. Fairchild's attempt [12] at establishing uniqueness for the problem (2), (3) was unsuccessful: the result and its proof were incorrect and the author had to publish an erratum [13].

12 The Bach action

In this section we consider the case when the weights appearing in formula (1) are

$$c_3 = 1 \quad \text{and} \quad c_j = 0, \quad \forall j \neq 3. \quad (42)$$

In this case our action becomes

$$S = \|\mathcal{W}\|^2, \quad (43)$$

where $\mathcal{W} = R^{(3)}$ is the Weyl curvature. The action (43) is called the *Bach action*. If one assumes the connection to be Levi-Civita and varies (43) with respect to the metric then the resulting Euler–Lagrange equation is the classical Bach equation [31]; see also [30] for an account of the modern state of the subject.

Our approach is to vary the metric and the connection independently, which leads to the metric-affine version (2), (3) of the Bach equation.

All the general results obtained in this paper apply to the Bach case (42). However, the peculiarity of this case is that any spacetime with zero Weyl curvature is a solution of the metric-affine Bach problem (2), (3); here we do not have to make any assumptions concerning the other irreducible pieces of curvature as we did in Section 8. Accordingly, the analysis of Riemannian solutions has to be modified (note that in the Bach case the first condition (38) is not satisfied). Equations (25), (26) are now equivalent to

$$\mathcal{W}^{\kappa\lambda\mu\nu}\mathcal{R}ic_{\kappa\mu} = 0, \quad (44)$$

$$\nabla_\lambda \mathcal{R}ic_{\kappa\mu} - \nabla_\kappa \mathcal{R}ic_{\lambda\mu} + \frac{1}{12}(g_{\kappa\mu}\partial_\lambda \mathcal{R} - g_{\lambda\mu}\partial_\kappa \mathcal{R}) \equiv 2\nabla_\nu \mathcal{W}_{\kappa\lambda\mu}{}^\nu = 0, \quad (45)$$

where we made use of the identity (34). Equation (45) does not imply that the scalar curvature is constant, so we lose the equation (29) which we previously derived under the assumption $c_1 + c_2 \neq 0$. Nevertheless, the system (44), (45) remains heavily overdetermined. It is natural to state the following

Conjecture 12.1 *In the Bach case (42) the only Riemannian solutions of our problem (2), (3) are conformally flat spaces and Einstein spaces.*

Though providing a rigorous proof of Conjecture 12.1 might be very difficult, one can easily check that it is true within the class of $2 + 2$ decomposable solutions.

Recall that the classical Bach equation has far more solutions than stated in Conjecture 12.1. Spacetimes which are conformally related to Einstein spaces are solutions, as are some non-trivial spacetimes found in [32] and [30]. Note that the paper [30] is based on the $2 + 2$ decomposition.

There is, of course, a fundamental difference in the role played by the Bach action in the classical (Riemannian) and metric-affine settings. In the classical setting the Bach action is very special in that it is constructed from the only conformally invariant irreducible piece of curvature, whereas in the metric-affine setting it loses its special status because all the irreducible pieces of curvature become conformally invariant; see also Remark 1.1.

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A Irreducible decomposition of curvature

We give below an overview of Appendix B.4 from [1], and present the results in a form suitable to our needs.

A Riemann curvature generated by a general affine connection has only one (anti)symmetry, namely,

$$R_{\kappa\lambda\mu\nu} = -R_{\kappa\lambda\nu\mu} . \quad (46)$$

For a fixed $x \in M$ we denote by \mathbf{R} the 96-dimensional vector space of real rank 4 tensors satisfying condition (46), and we equip \mathbf{R} with the natural indefinite inner product

$$(R, Q)_x := R^\kappa{}_{\lambda\mu\nu} Q^\lambda{}_{\kappa}{}^{\mu\nu} . \quad (47)$$

We have the orthogonal decomposition $\mathbf{R} = \mathbf{R}^+ \oplus \mathbf{R}^-$ where

$$\mathbf{R}^\pm = \{R \in \mathbf{R} | R_{\kappa\lambda\mu\nu} = \pm R_{\lambda\kappa\mu\nu}\} .$$

It is easy to see that $\dim \mathbf{R}^+ = 60$ and $\dim \mathbf{R}^- = 36$.

The subspaces \mathbf{R}^+ and \mathbf{R}^- decompose further into five and six irreducible subspaces respectively. We are mostly interested in \mathbf{R}^- as this is the vector space of curvatures generated by metric compatible connections, so what follows is a description of the irreducible subspaces of \mathbf{R}^- .

Put

$$(R^T)_{\kappa\lambda\mu\nu} := R_{\mu\nu\kappa\lambda} , \quad (48)$$

$$(*R)_{\kappa\lambda\mu\nu} := \frac{1}{2} \sqrt{|\det g|} \varepsilon^{\kappa'\lambda'}{}_{\kappa\lambda} R_{\kappa'\lambda'\mu\nu} , \quad (49)$$

$$(R^*)_{\kappa\lambda\mu\nu} := \frac{1}{2} \sqrt{|\det g|} R_{\kappa\lambda\mu'\nu'} \varepsilon^{\mu'\nu'}{}_{\mu\nu} . \quad (50)$$

The maps

$$R \rightarrow R^T , \quad (51)$$

$$R \rightarrow *R , \quad (52)$$

$$R \rightarrow R^* \quad (53)$$

are endomorphisms in \mathbf{R}^- . We call them *transposition*, *left Hodge star* and *right Hodge star* respectively. The left Hodge star acts on the internal (Lie algebra) indices of curvature, whereas the right Hodge star acts on the external ones and is the Hodge star used in abstract Yang–Mills theory.

The eigenvalues of the map (51) are ± 1 , whereas the maps (52) and (53) have no eigenvalues at all (as we are working in the real setting $\pm i$ are not eigenvalues). This impediment is overcome by working with the map

$$R \rightarrow *R^* \quad (54)$$

rather than with the maps (52) and (53) separately. Here

$$*R^* := (*R)^* = *(R^*) , \quad (55)$$

and the order of operations does not matter because the maps (52) and (53) commute. We call the endomorphism (54) the *double duality* map. Its eigenvalues are ± 1 .

Clearly, the maps (51) and (54) commute and square to the identity, so $\mathbf{R}^- = \bigoplus_{a,b=\pm} \mathbf{R}_{ab}^-$ where

$$\mathbf{R}_{ab}^- = \{R \in \mathbf{R}^- | R^T = aR, {}^*R^* = bR\}.$$

The maps (51) and (54) are formally self-adjoint with respect to the inner product (47) so the subspaces \mathbf{R}_{++}^- , \mathbf{R}_{+-}^- , \mathbf{R}_{-+}^- , and \mathbf{R}_{--}^- are mutually orthogonal. Their dimensions turn out to be 9, 12, 9, and 6 respectively.

For a Riemann curvature $R \in \mathbf{R}_{++}^-$ the corresponding Ricci curvature is symmetric trace free, and it completely determines R itself according to the formula

$$R_{\kappa\lambda\mu\nu} = \frac{1}{2}(g_{\kappa\mu}Ric_{\lambda\nu} - g_{\lambda\mu}Ric_{\kappa\nu} - g_{\kappa\nu}Ric_{\lambda\mu} + g_{\lambda\nu}Ric_{\kappa\mu}).$$

For a Riemann curvature $R \in \mathbf{R}_{--}^-$ the corresponding Ricci curvature is anti-symmetric, and it completely determines R itself according to the same formula. The subspace \mathbf{R}_{-+}^- is the image of \mathbf{R}_{++}^- under either of the maps (52) or (53). We see that each of the subspaces \mathbf{R}_{++}^- , \mathbf{R}_{-+}^- , \mathbf{R}_{--}^- is equivalent to a space of real rank 2 tensors, either symmetric trace free or antisymmetric. Therefore these three subspaces are irreducible.

The only subspace which decomposes further is \mathbf{R}_{+-}^- :

$$\mathbf{R}_{+-}^- = \mathbf{R}_{\text{scalar}} \oplus \mathbf{R}_{\text{Weyl}} \oplus \mathbf{R}_{\text{pseudoscalar}}.$$

Here $\mathbf{R}_{\text{scalar}}$ and $\mathbf{R}_{\text{pseudoscalar}}$ are the 1-dimensional spaces of real Riemann curvatures $R_{\kappa\lambda\mu\nu}$ proportional to $g_{\kappa\mu}g_{\lambda\nu} - g_{\lambda\mu}g_{\kappa\nu}$ and $\varepsilon_{\kappa\lambda\mu\nu}$ respectively, and \mathbf{R}_{Weyl} is their 10-dimensional orthogonal complement.

The decomposition described above assumes curvature to be real and metric to be Lorentzian. If curvature is complex or if $\det g > 0$ then the decomposition is somewhat different. In particular, the subspaces \mathbf{R}_{Weyl} and \mathbf{R}_{--}^- decompose further into eigenspaces of the Hodge star (left or right).

In order to simplify notation in the main text we will denote the subspaces

$$\mathbf{R}_{++}^-, \quad \mathbf{R}_{\text{scalar}}, \quad \mathbf{R}_{\text{Weyl}}, \quad \mathbf{R}_{\text{pseudoscalar}}, \quad \mathbf{R}_{-+}^-, \quad \mathbf{R}_{--}^-$$

by $\mathbf{R}^{(j)}$, $j = 1, \dots, 6$, respectively, and the five subspaces of \mathbf{R}^+ by $\mathbf{R}^{(j)}$, $j = 7, \dots, 11$.

Thus, at each point $x \in M$ the vector space of Riemann curvatures decomposes as $\mathbf{R} = \mathbf{R}^{(1)} \oplus \dots \oplus \mathbf{R}^{(11)}$. Consequently, a Riemann curvature R can be uniquely written as $R = R^{(1)} + \dots + R^{(11)}$ where $R^{(j)} \in \mathbf{R}^{(j)}$, $j = 1, \dots, 11$, are

its irreducible pieces. The explicit formulae for the first six pieces are

$$\begin{aligned}
R^{(1)}{}_{\kappa\lambda\mu\nu} &= \frac{1}{2}(g_{\kappa\mu}\overline{Ric}_{\lambda\nu} - g_{\lambda\mu}\overline{Ric}_{\kappa\nu} - g_{\kappa\nu}\overline{Ric}_{\lambda\mu} + g_{\lambda\nu}\overline{Ric}_{\kappa\mu}), \\
R^{(2)}{}_{\kappa\lambda\mu\nu} &= \frac{1}{12}(g_{\kappa\mu}g_{\lambda\nu} - g_{\lambda\mu}g_{\kappa\nu})\mathcal{R}, \\
R^{(3)} &= \overline{R} - R^{(1)} - R^{(2)} - R^{(4)}, \\
R^{(4)}{}_{\kappa\lambda\mu\nu} &= -\frac{1}{24}\sqrt{|\det g|} \varepsilon_{\kappa\lambda\mu\nu}\tilde{\mathcal{R}}, \\
R^{(5)} &= \widehat{R} - R^{(6)}, \\
R^{(6)}{}_{\kappa\lambda\mu\nu} &= \frac{1}{2}(g_{\kappa\mu}\widehat{Ric}_{\lambda\nu} - g_{\lambda\mu}\widehat{Ric}_{\kappa\nu} - g_{\kappa\nu}\widehat{Ric}_{\lambda\mu} + g_{\lambda\nu}\widehat{Ric}_{\kappa\mu}),
\end{aligned}$$

where

$$\begin{aligned}
\overline{R}_{\kappa\lambda\mu\nu} &= \frac{1}{4}(R_{\kappa\lambda\mu\nu} - R_{\lambda\kappa\mu\nu} + R_{\mu\nu\kappa\lambda} - R_{\nu\mu\kappa\lambda}), \\
\widehat{R}_{\kappa\lambda\mu\nu} &= \frac{1}{4}(R_{\kappa\lambda\mu\nu} - R_{\lambda\kappa\mu\nu} - R_{\mu\nu\kappa\lambda} + R_{\nu\mu\kappa\lambda}), \\
\overline{Ric}_{\lambda\nu} &= \overline{R}{}^{\kappa}{}_{\lambda\kappa\nu}, \quad \mathcal{R} = \overline{Ric}{}^{\lambda}{}_{\lambda} = R^{\kappa\lambda}{}_{\kappa\lambda}, \quad \overline{Ric}_{\lambda\nu} = \overline{Ric}_{\lambda\nu} - \frac{1}{4}g_{\lambda\nu}\mathcal{R}, \\
\widehat{Ric}_{\lambda\nu} &= \widehat{R}{}^{\kappa}{}_{\lambda\kappa\nu}, \\
\tilde{\mathcal{R}} &= \sqrt{|\det g|} \varepsilon^{\kappa\lambda\mu\nu}\overline{R}_{\kappa\lambda\mu\nu} = \sqrt{|\det g|} \varepsilon^{\kappa\lambda\mu\nu}R_{\kappa\lambda\mu\nu}.
\end{aligned}$$

Of course, in the Riemannian case curvature has only three irreducible pieces, namely, $R^{(1)}$, $R^{(2)}$, and $R^{(3)}$.

B Irreducible decomposition of torsion

According to Appendix B.2 from [1] the irreducible pieces of torsion are

$$T^{(1)} = T - T^{(2)} - T^{(3)}, \quad (56)$$

$$T^{(2)}{}_{\lambda\mu\nu} = g_{\lambda\mu}v_{\nu} - g_{\lambda\nu}v_{\mu}, \quad (57)$$

$$T^{(3)} = *w, \quad (58)$$

where

$$v_{\nu} = \frac{1}{3}T^{\lambda}{}_{\lambda\nu}, \quad w_{\nu} = \frac{1}{6}\sqrt{|\det g|} T^{\kappa\lambda\mu} \varepsilon_{\kappa\lambda\mu\nu}. \quad (59)$$

The pieces $T^{(1)}$, $T^{(2)}$ and $T^{(3)}$ are called *tensor torsion*, *trace torsion* and *axial torsion* respectively.

We define the action of the Hodge star on torsions as

$$(*T)_{\lambda\mu\nu} := \frac{1}{2}\sqrt{|\det g|} T_{\lambda\mu'\nu'} \varepsilon^{\mu'\nu'}{}_{\mu\nu}. \quad (60)$$

The Hodge star maps tensor torsions to tensor torsions, trace to axial, and axial to trace:

$$(*T)^{(1)} = *(T^{(1)}), \quad (61)$$

$$(*T)^{(2)}_{\lambda\mu\nu} = g_{\lambda\mu}w_\nu - g_{\lambda\nu}w_\mu, \quad (62)$$

$$(*T)^{(3)} = -*v. \quad (63)$$

Note that the $*$ appearing in the RHS's of formulae (58) and (63) is the standard Hodge star (11) which should not be confused with the Hodge star on torsions (60).

The decomposition described above assumes torsion to be real and metric to be Lorentzian. If torsion is complex or if $\det g > 0$ then the subspace of tensor torsions decomposes further into eigenspaces of the Hodge star.

Substituting formulae (56)–(58) into formula (6), and formula (8) into formulae (59) we obtain the irreducible decomposition of contortion:

$$K^{(1)} = K - K^{(2)} - K^{(3)}, \quad (64)$$

$$K^{(2)}_{\lambda\mu\nu} = g_{\lambda\mu}v_\nu - g_{\nu\mu}v_\lambda, \quad (65)$$

$$K^{(3)} = \frac{1}{2} * w, \quad (66)$$

where

$$v_\nu = \frac{1}{3} K^\lambda{}_{\lambda\nu}, \quad w_\nu = \frac{1}{3} \sqrt{|\det g|} K^{\kappa\lambda\mu} \varepsilon_{\kappa\lambda\mu\nu}. \quad (67)$$

The irreducible pieces of torsion (56)–(58) and contortion (64)–(66) are related as

$$T^{(j)}_{\lambda\mu\nu} = K^{(j)}_{\mu\lambda\nu}, \quad j = 1, 2, \quad T^{(3)}_{\lambda\mu\nu} = 2K^{(3)}_{\lambda\mu\nu}$$

(note the order of indices).

C Spinor representation of Weyl curvature

Throughout this appendix we work in Minkowski space, see (18). We follow Section 17 of [33] in our spinor notation, and we use the Latin letters a, b, c, d for spinor indices; these run through the values 1, 2. The Pauli and Dirac matrices are

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\gamma^0 = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}, \quad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}, \quad j = 1, 2, 3.$$

We write rank 1 bispinors as columns

$$\psi = \begin{pmatrix} \xi^1 \\ \xi^2 \\ \eta_1 \\ \eta_2 \end{pmatrix}.$$

The Dirac conjugate of ψ is $\gamma^0 \overline{\psi}$, with the “overline” standing for complex conjugation.

Given a pair of tensor indices κ, λ let us consider the matrix $\gamma^0 \gamma^2 \gamma^\kappa \gamma^\lambda$ and write it in block form

$$\gamma^0 \gamma^2 \gamma^\kappa \gamma^\lambda = \begin{pmatrix} (\gamma^0 \gamma^2 \gamma^\kappa \gamma^\lambda)_{ab} & (\gamma^0 \gamma^2 \gamma^\kappa \gamma^\lambda)_{a\dot{b}} \\ (\gamma^0 \gamma^2 \gamma^\kappa \gamma^\lambda)^{\dot{a}}_b & (\gamma^0 \gamma^2 \gamma^\kappa \gamma^\lambda)^{\dot{a}\dot{b}} \end{pmatrix}. \quad (68)$$

The diagonal blocks in the RHS of formula (68) are symmetric for $\kappa \neq \lambda$ and antisymmetric for $\kappa = \lambda$. The off-diagonal blocks are zero for all κ, λ .

It is easy to see that the matrix $\gamma^0 \gamma^2$ represents a Lorentz invariant linear map from the complex vector space of rank 1 bispinors to the complex vector space of conjugate rank 1 bispinors. Consequently, for an arbitrary rank 2 tensor Q the matrix $\gamma^0 \gamma^2 \gamma^\kappa \gamma^\lambda Q_{\kappa\lambda}$ has the same mapping property. This explains the choice of spinor indices in formula (68).

Let us recall the spinor representation of an antisymmetric rank 2 tensor, see Section 19 of [33] and Section 7 of [17] for details. A complex antisymmetric rank 2 tensor F is equivalent to a symmetric rank 2 bispinor

$$\begin{pmatrix} \phi^{ab} \\ \chi_{\dot{a}\dot{b}} \end{pmatrix}, \quad (69)$$

the relationship between the two being

$$F^{\kappa\lambda} = (\gamma^0 \gamma^2 \gamma^\kappa \gamma^\lambda)_{ab} \phi^{ab} + (\gamma^0 \gamma^2 \gamma^\kappa \gamma^\lambda)^{\dot{a}\dot{b}} \chi_{\dot{a}\dot{b}}. \quad (70)$$

Note that $*F = iF$ if and only if $\chi = 0$, and $*F = -iF$ if and only if $\phi = 0$.

We say that a complex rank 4 tensor R is a Weyl curvature if it satisfies

$$R_{\kappa\lambda\mu\nu} = -R_{\lambda\kappa\mu\nu} = -R_{\kappa\lambda\nu\mu} = R_{\mu\nu\kappa\lambda}, \quad (71)$$

$$*R^* = -R, \quad (72)$$

$$R^{\kappa\lambda}{}_{\kappa\lambda} = 0, \quad (73)$$

$$\varepsilon^{\kappa\lambda\mu\nu} R_{\kappa\lambda\mu\nu} = 0. \quad (74)$$

Formulae (69), (70) allow us to give a spinor representation of Weyl curvature. Namely, a complex Weyl curvature is equivalent to a rank 4 bispinor

$$\begin{pmatrix} \zeta^{abcd} \\ \omega_{\dot{a}\dot{b}\dot{c}\dot{d}} \end{pmatrix} \quad (75)$$

such that

$$\zeta^{abcd} = \zeta^{bacd} = \zeta^{abdc} = \zeta^{cdab}, \quad (76)$$

$$\omega_{\dot{a}\dot{b}\dot{c}\dot{d}} = \omega_{\dot{b}\dot{a}\dot{c}\dot{d}} = \omega_{\dot{a}\dot{b}\dot{d}\dot{c}} = \omega_{\dot{c}\dot{d}\dot{a}\dot{b}}, \quad (77)$$

$$\zeta^{\dot{a}\dot{b}}_{\dot{a}\dot{b}} = 0, \quad (78)$$

$$\omega^{\dot{a}\dot{b}}_{\dot{a}\dot{b}} = 0. \quad (79)$$

Weyl curvature is expressed via the bispinor (75) as

$$R^{\kappa\lambda\mu\nu} = (\gamma^0\gamma^2\gamma^\kappa\gamma^\lambda)_{ab}\zeta^{abcd}(\gamma^0\gamma^2\gamma^\mu\gamma^\nu)_{cd} \\ + (\gamma^0\gamma^2\gamma^\kappa\gamma^\lambda)^{\dot{a}\dot{b}}\omega_{\dot{a}\dot{b}\dot{c}\dot{d}}(\gamma^0\gamma^2\gamma^\mu\gamma^\nu)^{\dot{c}\dot{d}}. \quad (80)$$

Note that the spinor conditions (78), (79) are needed to ensure the fulfillment of the tensor conditions (73), (74). Note also that ${}^*R = R^* = iR$ if and only if $\omega = 0$, and ${}^*R = R^* = -iR$ if and only if $\zeta = 0$.

We say that the complex Weyl curvature R factorizes if $R_{\kappa\lambda\mu\nu} = F_{\kappa\lambda}F_{\mu\nu}$ for some antisymmetric rank 2 tensor F . We say that the spinor ζ in the bispinor (75) factorizes if $\zeta^{abcd} = \xi^a\xi^b\xi^c\xi^d$ for some rank 1 spinor ξ^a . We say that the spinor ω in the bispinor (75) factorizes if $\omega_{\dot{a}\dot{b}\dot{c}\dot{d}} = \eta_{\dot{a}}\eta_{\dot{b}}\eta_{\dot{c}}\eta_{\dot{d}}$ for some rank 1 spinor $\eta_{\dot{a}}$. Examination of formulae (71)–(80) establishes the following

Lemma C.1 *A complex Weyl curvature factorizes if and only if one of the spinors in the bispinor (75) factorizes and the other is zero.*

We see that a factorized complex Weyl curvature is equivalent to a rank 1 spinor ξ^a or $\eta_{\dot{a}}$. This spinor is, effectively, the fourth root of curvature, and is determined uniquely up to multiplication by i^n , $n = 0, 1, 2, 3$.

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